MECHANICAL STRESSES IN A LONG CYLINDRICAL CURRENT-CARRYING SOLENOID

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Expressions for mechanical stresses and strains in a multilayer cylindrical solenoid carrying direct current are derived and analyzed. The results are valid for both an ordinary and a superconductive solenoid whose length is large compared with its transverse dimensions.

§1. The mechanical stresses which arise in a current carrying solenoid can result in its destruction. Hence, in designing solenoids capable of withstanding high-intensity magnetic fields it is necessary to know the distribution and magnitude of the maximum stresses in the coil. This can be done by solving the system of equilibrium equations

$$\partial\sigma_{ik} / \partial x_k + f_i = 0 \qquad (1.1)$$

under the appropriate boundary conditions. Here σ_{ik} is the mechanical stress tensor and f_i is the volume density of the ponderomotive forces acting on the material in the magnetic field.

The authors of works dealing with this problem (e. g., see [1]) simplify it either by neglecting certain components of the stress tensor or by neglecting the conditions of continuity of the displacement vector (the so-called St. Venant continuity conditions) which must be added to system (1.1) in order for its solutions to have physical meaning.

Correct solution of system (1.1) in the general case presents serious mathematical difficulties. In the present paper we shall obtain a solution for a multilayer cylindrical solenoid whose length is large compared with its transverse dimensions.

We assume that the solenoid can be regarded as a continuous medium. In the case of a wire solenoid this is permissible provided there are no shear strains and that only compressive stresses act in directions perpendicular to the coil windings. This assumption is also valid if we assume that the solenoid windings are glued together. The medium can be considered homogeneous if the conductor is thin (as compared with all the other dimensions of the solenoid) and if the coil is sufficiently tight (dense). In the range of small strains (the range in which Hooke's law holds) the elastic moduli of the wire in the transverse and longitudinal directions are equal. This enables us to consider the medium isotropic. Finally, since the thickness of the wire is small as compared with the linear dimensions of the solenoid,* it is also small as compared with the distances at which the magnetic field and ponderomotive force change markedly. Hence, the current distribution over the cross section of the conductor is negligible. One of the implications of this is that our results are also applicable to superconductive solenoids.

§2. Let us determine the components of the ponderomotive force f_i. We disregard the magnetic properties of the medium, i. e., we set the magnetic permeability μ equal to unity everywhere. The volume density of the forces acting on the medium in the magnetic field is then given by the formula [2]

$$\mathbf{f} = \frac{1}{c}\mathbf{j}\times\mathbf{H}, \qquad \mathbf{H} = \frac{1}{c}\int\frac{\mathbf{j}\times\mathbf{R}}{R^3}\,dV, \qquad \mathbf{j} = \mathbf{J}n^2 \, . \qquad (2.1)$$

Here j is the current density, H is the magnetic field intensity, R is the radius vector directed from dV to the point of observation, and n is the number of windings per unit length. For simplicity we assume that the current J is equal in all the windings.

In the cylindrical coordinate system (r, φ, z) we obtain the following expressions for the components of the force f:

$$f_r(r, z) = \frac{j^2}{c^2}\int\limits_{-l-z}^{l-z} d\zeta \int\limits_{b}^{B} r'\,dr' \int\limits_{0}^{2\pi} d\psi\, \frac{r' - r\cos\psi}{(\zeta^2+\rho^2)^{3/2}}$$

$$f_z(r, z) = \frac{j^2}{c^2}\int\limits_{-l-z}^{l-z} d\zeta \int\limits_{b}^{B} r'\,dr' \int\limits_{0}^{2\pi} d\psi\, \frac{\zeta\cos\psi}{(\zeta^2+\rho^2)^{3/2}}$$

$$\zeta = z' - z, \quad \psi = \varphi' - \varphi,$$

$$\rho^2 = r'^2 + r^2 - 2r'r\cos\psi \, . \qquad (2.2)$$

Here $2l$ is the length of the solenoid, B is its outside radius, and b is its inside radius. By virtue of the cylindrical symmetry of the problem the components f_r and f_z do not depend on the coordinate φ, and $f_\varphi = 0$.

Analysis of expressions (2.2) shows that the force component f_z which compresses the solenoid along the z-axis, is equal to zero in the middle cross section of the solenoid; it then increases monotonically with increasing z, slowly at first, and then very rapidly towards the ends of the solenoid, reaching its maximum at the end faces ($z = \pm l$). The force component f_r, directed everywhere along the radius, is maximum in the middle cross section of the solenoid, then decreases with increasing z, slowly at first, and rapidly as it approaches the ends. Since $f_z \sim H_r$ and $f_r \sim H_z$, it follows that for j = const the corresponding components of the magnetic field in a cylindrical solenoid depend on z in exactly the same way.

For a long solenoid ($B \ll l$) we can expand the integrands in expressions (2.2) in a series in the parameters $\rho/(l \pm z)$, assume that $\rho \ll l \pm z$, and carry out the

*The contribution of the neighboring windings to the magnetic field is negligibly small as compared with the contribution of the other windings whose magnetic field is approximately homogeneous.

appropriate integrations. This yields

$$f_r = \frac{j_2}{c^2}\int_b^B r'\,dr' \int_0^{2\pi} d\psi \frac{1}{\rho^2}(r' - r\cos\psi) \times$$

$$\times\left[\frac{l-z}{\sqrt{(l-z)^2+\rho^2}} + \frac{l+z}{\sqrt{(l+z)^2+\rho^2}}\right] \approx$$

$$\approx 4\pi\frac{j^2}{c^2}(B-r) - \frac{\pi}{3}\frac{j^2}{c^2}(B^3-b^3)\times$$

$$\times[(l-z)^{-2}+(l+z)^{-2}]+\ldots \qquad (2.3)$$

$$f_z = -\frac{j^2}{c^2}\int_b^B r'\,dr' \int_0^{2\pi} d\psi \cos\psi \times$$

$$\times\left[\frac{1}{\sqrt{(l-z)^2+\rho^2}} - \frac{1}{\sqrt{(l+z)^2+\rho^2}}\right] \approx$$

$$\approx -\frac{\pi}{3}\frac{j^2}{c^2}(B^3-b^3)\,r\,[(l-z)^{-3}-(l+z)^{-3}]+)\ldots. \qquad (2.4)$$

These expressions are clearly valid far away from the ends, i.e., for $|z| \ll l - B$. In the zeroth approximation, i.e., in the limiting case of an infinitely long solenoid, we have

$$f_r = 4\pi\frac{j^2}{c^2}(B-r), \qquad f_z = 0\,. \qquad (2.5)$$

§3. Now let us formulate the problem of elastic equilibrium of the solenoid under ponderomotive forces. Making use of Hooke's law for an isotropic body [3],

$$\sigma_{ik} = \frac{E}{1+\sigma}\left(u_{ik} + \frac{\sigma}{1-2\sigma}u_{jj}\delta_{ik}\right) \qquad (3.1)$$

(E is Young's modulus, σ is the Poisson coefficient) and the expression relating* the strain tensor u_{ik} to the displacement vector u_i,

$$u_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \qquad (3.2)$$

we can rewrite equilibrium Eq. (1.1) as [3]

$$\Delta\mathbf{u} + \frac{1}{1-2\sigma}\,\mathrm{grad\,div}\,\mathbf{u} = -\frac{2(1-\sigma)}{E}\,\mathbf{f}. \qquad (3.3)$$

Since the problem is cylindrically symmetric, u does not depend on the coordinate φ, and $u_\varphi \equiv 0$. Hence, it is convenient to write Eq. (3 3) in cylindrical coordinates,

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{(1-2\sigma)}{2(1-\sigma)}\frac{\partial^2}{\partial z^2}\right]u_r +$$

$$+\frac{1}{2(1-\sigma)}\frac{\partial^2}{\partial r\,\partial z}u_z = -\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)E}f_r,$$

$$\left[\frac{\partial^2}{\partial z^2} + \frac{(1-2\sigma)}{2(1-\sigma)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial}{\partial r}\right]u_z +$$

$$+\frac{1}{2(1-\sigma)}\left(\frac{\partial}{\partial r}+\frac{1}{r}\right)\frac{\partial}{\partial z}u_r =$$

$$= -\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)E}f_z\,. \qquad (3.4)$$

*The St. Venant continuity condition is fulfilled automatically in this case.

The following boundary conditions must be fulfilled at the ends and side surfaces of the solenoid:

At the free ends of the solenoid

$$\sigma_{rz} = \sigma_{\varphi z} = \sigma_{zz} = 0 \quad \text{for} \quad z = \pm l\,. \qquad (3.5)$$

At the free side surfaces of the solenoid

$$\sigma_{rr} = \sigma_{r\varphi} = \sigma_{rz} = 0 \quad \text{for} \quad r = b \quad \text{or} \quad r = B\,. \qquad (3.6)$$

At fixed side surfaces of the solenoid

$$u_r = 0 \quad \text{for} \quad r = b \quad \text{or} \quad r = B\,. \qquad (3.7)$$

§4. In order to find a solution valid far from the ends of a long solenoid we make use of the St. Venant principle (which should not be confused with the St. Venant continuity condition) according to which the distribution of strains and stresses far from the points of application of forces does not depend on the force distribution, but only on the total force and on the total moment of forces [4]. We therefore proceed as follows. We isolate the largest portion of the solenoid at which expressions for the volume forces Eqs. (2.3) and (2.4) are valid. We substitute these expressions into the right sides of Eqs. (3.4) and replace the forces acting outside this isolated portion by the sum forces applied to its boundaries. The solution of the problem thus formulated is, by the St. Venant principle, close to the solution of the initial problem far away from the boundaries of the domain, i.e., in the middle cross section of the solenoid.

Let us determine the sum force acting along the z-axis over the portion of the solenoid extending from the cross section z to l To do this we integrate the expression for f_z in Eq. (2.2) over z from z to l. This can be done, as in integrating Eqs. (2.3) and (2.4), by expanding the integrand in a series in the parameters $\rho/(l \pm z)$,

$$F_z(r,z) = \int_z^l f_z\,dz = -\frac{j^2}{c^2}\int_b^B r'\,dr'\times$$

$$\times\int_0^{2\pi} d\psi\cos\psi\left(\operatorname{arc\,sh}\frac{l-z}{\rho} +\right.$$

$$\left.+\operatorname{arc\,sh}\frac{l+z}{\rho} - \operatorname{arc\,sh}\frac{2l}{\rho}\right) \approx$$

$$\approx -\frac{\pi}{3}\frac{j^2}{c^2}\left(-2r^2 + 3Br - \frac{b^3}{r}\right) +$$

$$+\frac{\pi}{6}\frac{j^2}{c^2}(B^3-b^3)\,r\,[(l-z)^{-2} +$$

$$+(l+z)^{-2} - (2l)^{-2}] + \ldots\,. \qquad (4.1)$$

Integrating F_z over the cross section of the solenoid, we obtain the sum force Φ_z,

$$\Phi_z(z) = \int_b^B r\,dr\int_0^{2\pi} d\varphi\,F_z =$$

$$= -\frac{\pi^2}{3}\frac{j^2}{c^2}(B-b)^2(B^2+2Bb+3b^2)+ \qquad (4.2)$$

$$+\frac{\pi^2}{9}\frac{j^2}{c^2}(B^3-b^3)^2[(l-z)^{-2}+ \\ +(l+z)^{-2}-(2l)^{-2}]+\ldots \tag{4.2 (cont'd)}$$

The sum force due to f_r is equal to zero.

Boundary condition (3.5) for σ_{zz} must be replaced by the condition

$$\int_b^B r\,dr\int_0^{2\pi} d\varphi\,\sigma_{zz}|_{z=z^*}=\Phi_z(z^*), \tag{4.3}$$

where $z = z^*$ is the cross section bounding the isolated portion of the solenoid. The inequalities $B \ll z^* \ll l - B$ must be fulfilled here—something which is impossible in the case of a long solenoid ($B \ll l$). The first inequality is necessary to the existence of a domain in which the St. Venant principle holds; the second inequality is necessary to the validity of expressions (2.3), (2.4), and (4.2).

§5. Let us find the solution in the zeroth approximation, i. e., the solution valid for an infinitely long solenoid. We substitute expressions (2.5) into the right sides of Eqs. (3.4) and the first term of series (4.2) into boundary condition (4.3). The solution which is sufficiently general to satisfy all the boundary conditions (4.3), (3.5), (3.6), or (3.7) then turns out to be

$$u_r=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^3}{(1-\sigma)E}\Big\{(1+\sigma)(1-2\sigma)(3t^3-8t^2)+ \\ +[(1-\sigma)\alpha-\sigma\gamma]t+(1+\sigma)\beta\frac{x^2}{t}\Big\}$$

$$u_z=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2z}{(1-\sigma)E}(-2\sigma\alpha+\gamma)\cdot \tag{5.1}$$

Here and below we make use of the dimensionless quantities

$$t=r/B,\qquad x=b/B. \tag{5.2}$$

The first two terms of u_r in Eq. (5.1) represent a particular solution of the inhomogeneous equation. The constants α, β, γ to be determined from the boundary conditions are chosen in such a way that the components σ_{ik} which are of the greatest interest can be expressed in simple form.

For the components of the strain tensor u_{ik}, solution (5.1) yields the following expressions in cylindrical coordinates:

$$u_{rr}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)E}\Big\{(1+\sigma)(1-2\sigma)(9t^2-16t)+ \\ +(1-\sigma)\alpha-\sigma\gamma-(1+\sigma)\beta\frac{x^2}{t^2}\Big\},$$

$$u_{\varphi\varphi}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)E}\Big\{(1+\sigma)(1-2\sigma)(3t^2-8t)+ \\ +(1-\sigma)\alpha-\sigma\gamma+(1+\sigma)\beta\frac{x^2}{t^2}\Big\},$$

$$u_{zz}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)E}(-2\sigma\alpha+\gamma),$$

$$u_{r\varphi}=u_{rz}=u_{\varphi z}=0. \tag{5.3}$$

Making use of formulas (3.1) and (5.3), we can find the components of the stress tensor

$$\sigma_{rr}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)}\times \\ \times\Big\{(9-6\sigma)t^2-(16-8\sigma)t+\alpha-\beta\frac{x^2}{t^2}\Big\},$$

$$\sigma_{\varphi\varphi}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)}\times \\ \times\Big\{(3+6\sigma)t^2-(8+8\sigma)t+\alpha+\beta\frac{x^2}{t^2}\Big\},$$

$$\sigma_{zz}=\frac{\pi}{6}\frac{j^2}{c^2}\frac{B^2}{(1-\sigma)}\{12\sigma t^2-24\sigma t+\gamma\}$$

$$\sigma_{r\varphi}=\sigma_{rz}=\sigma_{\varphi z}=0\cdot \tag{5.4}$$

From boundary condition (4.3) we obtain

$$\gamma=(1+x)^{-1}[(-2+12\sigma)\times \\ \times(1+x+x^2)+(6-12\sigma)x^3]. \tag{5.5}$$

Boundary conditions (3.6) and (3.7) yield the following values for the coefficients α and β.

In the case of a free outside and a free inside surface of the solenoid,

$$\alpha=7-2\sigma+\beta x^2, \\ \beta=(1+x)^{-1}[(7-2\sigma)-(9-6\sigma)x]\cdot \tag{5.6}$$

In the case of a free outside and a fixed inside surface,

$$\alpha=7-2\sigma+\beta x^2, \\ \beta=(1+x)^{-1}[(1+\sigma)+ \\ +(1-\sigma)x^2]^{-1}[(-7+7\sigma+10\sigma^2)+ \\ +(1-\sigma-6\sigma^2)x+(1-\sigma)\times \\ \times(5-2\sigma)x^2-(1-\sigma)(3-6\sigma)x^3]. \tag{5.7}$$

In the case of a fixed outside and a free inside surface,

$$\alpha=(16-8\sigma)x-(9-6\sigma)x^2+\beta, \\ \beta=(1+x)^{-1}[(1-\sigma)+ \\ +(1+\sigma)x^2]^{-1}[(1-\sigma)(5-2\sigma)- \\ -(1-\sigma)(11-6\sigma)x- \\ -(7-7\sigma-10\sigma^2)x^2+(9-9\sigma-6\sigma^2)x^3]. \tag{5.8}$$

In the case of a fixed outside and a fixed inside surface,

$$\alpha=(1+x)^{-1}[(5-2\sigma)(1+x+x^2)-(3-6\sigma)x^3], \\ \beta=(1+x)^{-1}(1-2\sigma)(-5+3x)\cdot \tag{5.9}$$

§6. To determine the solution in the next (first) approximation and thereby determine the degree of accuracy of solutions (5.1)–(5.9) above, we substitute expressions (2.3), (2.4) into Eqs. (3.4) and expression (4.2) into boundary condition (4.3). This yields the following terms which must be added to the zeroth approximation (5.1) for u_i:

$$-\frac{\pi}{9}\frac{j^2}{c^2}\frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)E}B^3(1-x^3)t^2\times$$

$$\times\left[\frac{B^3}{(l-z)^2}+\frac{B^2}{(l+z)^2}\right] \quad \text{for} \quad u_r, \tag{6.1}$$

$$\frac{\pi}{9}\frac{j^2}{c^2}\frac{(1+\sigma)}{E}B^3(1-x^3)t^3\times$$

$$\times\left[\frac{B^3}{(l-z)^3}-\frac{B^3}{(l+z)^3}\right] \quad \text{for} \quad u_z. \tag{6.2}$$

Appropriate terms must also be added to solutions (5.3), (5.4) But boundary conditions (3.6), (3.7) at the side surfaces of the solenoid can now be satisfied only for $z = 0$. In other words, the above solution (6.1), (6.2) is valid only in the middle cross section of the solenoid. However, the domain surrounding $z = 0$ is most significant, since it is here that the stresses have their extremal values. Without determining the coefficients α, β, γ, we merely note that the corrections for the zeroth solution are of the order B^2/l^2 and reduce the absolute values of the stresses. Thus, solutions (5.1)–(5.9) represent the upper limit for solenoids of finite length.

§7. The above results also apply to superconductive solenoids. In fact, the materials of which such solenoids are made are superconductors of the second kind. These are characterized by having two critical fields H_{c_1} and H_{c_2}. Fields $H < H_{c_1}$ do not penetrate into the superconductor; fields $H > H_{c_2}$ destroy superconductivity. The magnetic permeability μ of the superconductor therefore depends on the intensity of the magnetic field: $\mu = 0$ for $H < H_{c_1}$; as the field increases beyond H_{c_1}, the magnetic permeability increases sharply and quickly approaches unity, remaining at $\mu \approx 1$ all the way to $H = H_{c_2}$ [5].

In a long solenoid the field H_z varies approximately linearly over the radius (see (2.3)), vanishing at the outside radius. Hence, μ is different from unity in some domain close to the outer boundary of the solenoid. Formulas (2.1) are not valid here, and the above analysis is, strictly speaking, invalid. However, it is usually the case that $H_{c_1} \ll H_{c_2}$. The above domain is therefore narrow, its linear dimensions along the radius being $\Delta \sim BH_{c_1}/H_{c_2} \ll B$. Moreover, the body force Eq. (2.3) in this domain is also small, $f_r \sim 4\pi\Delta j^2/c^2$. Hence, allowance for the fact that μ vanishes and that there is a μ gradient in the domain in question has no significant effect on the final results, and the stress distribution in a long superconductive solenoid can be determined from formulas (5.1)–(5.9).

§8. In investigating the above solutions we are interested primarily in the extremal values of the stress tensor components and in their localization. Comparison of the values of σ_{ik} in various solenoids is best carried out for the same value of the magnetic field intensity H_0 at the solenoid axis. From formulas (2.1) and (2.5) we find that

$$H_0 = 4\pi\frac{j}{c}B(1-x). \tag{8.1}$$

The coefficient in front of the braces in expressions (5.4) for σ_{ik} can now be expressed as $H_0^2[96\pi(1-\sigma)(1-x)^2]^{-1}$. The value of the Poisson coefficient σ is determined by the material of which the solenoid is made. For most materials σ is close to 1/3. We shall therefore make frequent use of the value $\sigma = 1/3$. For these H_0 and σ, expressions (5.4) depend on the parameter x (see (5.2)) and on the choice of boundary conditions.

Analysis of expressions (5.4)–(5.8) shows that the condition of a fixed outside surface of a solenoid (cases (5.8) and (5.9)) is more advantageous (advantageous conditions are those for which the maximum stresses are minimal) than the condition of a free outside surface (cases (5.6) and (5.7)). Hence, the component $\sigma_{\varphi\varphi}$ has its largest positive values (tensile stresses are positive, compressive stresses are negative) for all t and x in case (5.6), while the component σ_{rr} has its largest positive values for all t and x in case (5.7). These components (henceforth all values of σ_{ik} will be given in units of $H_0^2/96\pi$) assume their maximum values at the inside surface of the solenoid for $t = x$,

$$\sigma_{\varphi\varphi} = 2(1-\sigma)^{-1}(1-x^2)^{-1}\times$$

$$\times[(7-2\sigma)+(2-4\sigma)x+(3-6\sigma)x^2], \tag{8.2}$$

$$\sigma_{rr} = 2[(1+\sigma)+(1-\sigma)x^2]^{-1}[7+6\sigma+2x+3x^2]. \tag{8.3}$$

The values of these maxima are minimal for $x = 0$ and are equal to $\sigma_{\varphi\varphi} = (14-4\sigma)/(1-\sigma)$ or 19 (for $\sigma = 1/3$) and to $\sigma_{rr} = (14+12\sigma)/(1+\sigma)$ or 13.5 (for $\sigma = 1/3$). With increasing x the values of Eqs. (8.2) and (8.3) increase, and in the limiting case $\Delta \ll 1$ for $x = 1-\Delta$ we have $\sigma_{\varphi\varphi} = 12/\Delta$, $\sigma_{rr} = 12+6\sigma$ or 14 ($\sigma = 1/3$).

Let us analyze the more advantageous cases (5.8) and (5.9) in more detail. The component $\sigma_{\varphi\varphi}$ in case (5.8) decreases monotonically with increasing t, so that the extremal values occur at the inside and outside surfaces of the solenoid. At the inner surface for $t = x$ we have

$$\sigma_{\varphi\varphi} = 2(1-\sigma)^{-1}(1-x^2)^{-1}[(1-\sigma)+$$

$$+(1+\sigma)x^2]^{-1}[(1-\sigma)(5-2\sigma)-$$

$$-(1-\sigma)(2+4\sigma)x-(8-2\sigma-16\sigma^2)x^2+$$

$$+(1+\sigma)(2-4\sigma)x^3+(1+\sigma)(3-6\sigma)x^4]. \tag{8.4}$$

For $x = 0$ this value is maximal and equal to $(10-4\sigma)/(1-\sigma)$ or 13 ($\sigma = 1/3$). Expression (8.4) decreases with increasing x, passing through zero for some $x = x_0$. For $\sigma = 1/3$ we have $x_0 \approx 0.59$.

At the outside surface of the solenoid for $t = 1$ we have

$$\sigma_{\varphi\varphi} = -4\sigma(1-\sigma)^{-1}(1-x^2)^{-1}[(1-\sigma)+(1+\sigma)x^2]^{-1}\times$$

$$\times[(1-\sigma)+(2-2\sigma)x+(6-5\sigma)x^2+2\sigma x^3-3x^4]. \tag{8.5}$$

For $x = 0$ this value is equal to $-4\sigma/(1-\sigma)$ or -2 ($\sigma = 1/3$), becoming still more negative with increasing x. In the limiting case $\Delta \ll 1$ for $x = 1-\Delta$ we have $\sigma_{\varphi\varphi} = -6\sigma/\Delta$.

In case (5.9) the component $\sigma_{\varphi\varphi}$ has a positive maximum localized near the inside surface of the solenoid. For $x = 0$ its value is $(5-2\sigma)/(1-\sigma)$ or 6.5 ($\sigma = 1/3$). With increasing x the maximum diminishes and vanishes for some $x = x_0$. For $\sigma = 1/3$ we have $x_0 \approx 0.28$. At the inside surface of the solenoid for $t = x$ we have

$$\sigma_{\varphi\varphi} = 4\sigma(1-\sigma)^{-1}(1-x^2)^{-1}(2-2x-3x^2). \tag{8.6}$$

For $x = 0$ this value is $8\sigma/(1-\sigma)$ or 4 ($\sigma = 1/3$). With increasing x expression (8.6) diminishes, passing through zero for $x = x_0 = (7-1)^{1/2}/3 \approx 0.55$.

At the outside surface of the solenoid for $t = 1$ we have

$$\sigma_{\varphi\varphi} = -4\sigma(1-\sigma)^{-1}(1-x^2)^{-1}(1+2x). \tag{8.7}$$

For $x = 0$ this value is equal to $-4\sigma/(1-\sigma)$ or -2 ($\sigma = 1/3$). With increasing x expression (8.7) diminishes, and in the limiting case $\Delta \ll 1$ for $x = 1-\Delta$ we have $\sigma_{\varphi\varphi} = -6\sigma/(1-\sigma)\Delta$ or $-3/\Delta$ ($\sigma = 1/3$).

The component σ_{rr} in case (5.8) vanishes for $t = x$ and has two extrema in the range $x < t < 1$. The first of these extrema is a positive maximum. For $x = 0$ it is localized near the inside surface and is equal to $(5-2\sigma)/(1-\sigma)$ or 6.5 ($\sigma = 1/3$); with increasing x the maximum shifts toward large t and diminishes, vanishing for some $x = x_0$. For $\sigma = 1/3$ we have $x_0 \approx 0.26$. The second extremum is a negative minimum. It is localized near the inside surface of the solenoid and its values are practically equal to the values of σ_{rr} exactly on the surface for t = 1,

$$\sigma_{rr} = -[(1-\sigma)+(1+\sigma)x^2]^{-1}[2+4x+(18+12\sigma)x^2]. \quad (8.8)$$

As x changes from 0 to 1 this value changes from −3 to −14 ($\sigma = 1/3$).

In case (5.9) the component σ_{rr} behaves like $\sigma_{\varphi\varphi}$ in case (5.8). At the inside surface of the solenoid for t = x we have

$$\sigma_{rr} = 2(1-\sigma)^{-1}(1-x^2)^{-1}[5-6\sigma-2x-3x^2]. \quad (8.9)$$

For x = 0 this value is maximal and equal to $(10 - 12\sigma)/(1 - \sigma)$ or 9 ($\sigma = 1/3$). With increasing x expression (8.9) diminishes, passing through zero for $x = x_0 = ((16 - 18\sigma)^{1/2} - 1)/3$ or $x_0 \approx 0.72$ ($\sigma = 1/3$).

At the outside surface of the solenoid for t = 1 we have

$$\sigma_{rr} = -2(1-\sigma)^{-1}(1-x^2)^{-1}[1+2x-(3-6\sigma)x^2]. \quad (8.10)$$

For x = 0 this value is equal to $-2/(1 - \sigma)$ or −3 ($\sigma = 1/3$). In the limiting case $\Delta \ll 1$ for $x = 1 - \Delta$ we have $\sigma_{rr} = -6\sigma/(1 - \sigma)\Delta$ or $-3/\Delta$ ($\sigma = 1/3$).

The component σ_{zz} does not depend on the choice of boundary conditions (see (5.5)). It assumes its extremal values at the inside surface of the solenoid for t = x,

$$\sigma_{zz} = 2(1-\sigma)^{-1}(1-x^2)^{-1}[-1+6\sigma-2x-3x^2]. \quad (8.11)$$

and at the outside surface for t = 1,

$$\sigma_{zz} = -2(1-\sigma)^{-1}(1-x^2)^{-1}[1+2x-(3-6\sigma)x^2]. \quad (8.12)$$

Value (8.11) is maximal for x = 0 and equal to $(-2 + 12\sigma)/(1 + \sigma)$ or 3 ($\sigma = 1/3$). With increasing x it diminishes, passing through zero for $x = x_0 = ((-2 + 18\sigma)^{1/2} - 1)/3$ or $x_0 = 1/3$ ($\sigma = 1/3$). The value of Eq. (8.12) for x = 0 is equal to $-2/(1 - \sigma)$ or −3 ($\sigma = 1/3$). With increasing x it diminishes, and is equal to $-6/\Delta$ in the limiting case for $x = 1 - \Delta$ ($\Delta \ll 1$).

Thus, for a solenoid with a fixed outside surface the maximum tensile stresses are localized near the inside surface. They decrease with increasing x and vanish for some $x = x_0$. The maximum compressive stresses are localized near the outside surface, and increase in absolute value with increasing x. The cases of a free Eq. (5.8) and fixed Eq. (5.9) inside surface are competitive, and the advantage of one over the other is determined by the choice of optimal values of x. This choice in turn depends on the difference in the effects of compressive and tensile stresses.

We must also note that the components σ_{rr} and σ_{zz} are positive, i.e., tensile, near the inside surface over a certain range of values of x from 0 to some x_0. The windings of a wire solenoid will separate here (if they are not glued together). The solenoid can then no longer be considered a continuous medium, and the above solutions are not strictly applicable. The formulation of the problem must be altered.

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